



Mathematical Model And Stability Analysis Of Typhoid Fever

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ABSTRACT

Typhoid fever is primarily transmitted through bacterial known as *Salmonella typhi* organism, a deterministic typhoid fever model is formulated by incorporating vital dynamic, standard incidence and control measures. The effective basic reproduction number R^c is computed by using next generation approach. The steady states are investigated and found to be stable. It is proved that the disease free-equilibrium is locally and globally stable when $R^c < 1$ and unstable when $R^c > 1$.

Keywords: Nonlinear ODE, Residence time, Continuous stirred tank reactor (CSTR)

INTRODUCTION

Typhoid fever in human beings is the most common bacterial infection due to different species of salmonella (WHO, 2003). "It is an endemic infections disease". According to (WHO, 2003) *Salmonella paratyphi* A, B,C and D are the most common species of salmonella that cause typhoid fever. It is transmitted by ingestion of drinking or eating food or water that has been contaminated with the faces of an Infected person or carrier (WHO, 2007). Poor hygiene and poor sanitation are the risk factors (3). Poor appetite, fatigue, sustained fever, severe headache and vomiting were the signs and symptoms of the typhoid fever. It takes about 7-14 days of incubation period (Jenkins C *et al.*, 2009).

Anwar *et al.*, (2014) said that "treatment is based on antibiotic susceptibility of the patient blood culture. The oral chloramphenicol amoxicillin may be used if the strain is sensitive. The chronic carrier state may be eradicated using oral therapy using ciprofloxacin or norfloxacin. Mult-drug resistance strains of salmonella typhi are increasingly common worldwide which made treatment by antibiotics more different and costly".

According to WHO (2003); "Many developing countries, the public health goals that can help in preventing and controlling the speed of typhoid fever infection through the safe drinking water, improved sanitation and adequate medical care may be difficult to achieve. Health education is paramount to raise public awareness and induce behavior

change" Kariuki (2008) said that an estimated 20 million cases of typhoid fever of which 200,000 were killed annually. It is believed that vaccinating the high-risk populations is the best way in controlling typhoid fever infection. Oral and injectable vaccines are the two (2) types of typhoid fever disease and therefore they are not effective 100%. "If one acquires drug-resistant strain of typhoid fever and is not treated with effective antibiotics, a serious and prolonged illness may result"

In the last decade, many researchers have proposed SIR-B epidemic model to study the transmission dynamic of typhoid fever. For instance, see Cvjetanovic *et al.*, (1971) Cook. (2010), Pitzer.*et al.*, (2014), Adetunde, (2008), Moffat *et al* (2014), Steady *et al.*, (2014), Adebaye and Haruna (2015), Khan *et al.*, (2015), Omame *et al.*, (2015), Nthiiri (2016), Getacheow *et al.*, (2017) and Edward (2017).

Getachew *et al.*, (2017) proposed and analyzed a compartmental nonlinear deterministic mathematical model for the typhoid fever outbreak that incorporates the optimal control with cost-effective strategies. There result revealed that a combination of prevention and treatment is the best cost-effective strategy to eradicate the disease. In a similar development, Edward (2017) formulated a mathematical model for typhoid fever to assess

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the effects of education campaigns, vaccination and treatment on controlling the transmission dynamics of typhoid fever in a community. The funding showed that when each of the controls increased it leads to a decrease in outbreak.

Nthiiri *et al.*, (2016) developed a mathematical model that incorporates protection against the infection but ignored the Recovered class and Bacterial population. In the present work, we therefore extend the model developed by Nthiiri the model by incorporating recovered class and the bacterial population and also a control measures to prevent the infections.

MODEL DEVELOPMENT

The model considers human population as well as bacterial population (B). The human population is divided in to four (4) compartments. Susceptible individuals S(t) is generated from constant recruitment of individuals through birth at the rate bN and arrival of recovered individuals due to the waning of temporary immunity at the rate ω . The Susceptible individuals will acquire typhoid fever that caused the bacteria when they ingest foods or waters that has been contaminated by Salmonella bacteria and move to the infected class at a rate $\frac{a(1-c_w)B}{K+B}$, where a is the Salmonella

ingestion rate, c_w is rate of compliance with water hygiene, K is Concentration of salmonella bacteria in foods and waters and B

$\frac{B}{K+B}$ is the probability of individuals that consume foods and drinks contaminated with typhoid causing bacteria. Infected individuals is decrease as a result of treatment, death occurring due to natural caused and disease induce death rate at θ, μ, δ_1 respectively.

Treated individuals grows treatment at the rate θ and reduces as a result death naturally and

disease –oriented death at the rate μ and δ_2 respectively.

Conclusion

In this paper, a brief introduction was given about the reactor and its behavior as regards to mixing, a schematic diagram was also drawn to enable one understand the nature of movements of the reaction. Then the dynamics of the reaction model was carried out were results were drawn based on the analysis. It is obvious that all concentration will reach an equilibrium point only that this will happen at various times as shown in the analysis. This has really proved the characteristic of a continuous flow stirred tank reactor as concentration in the tank is equal to the out flow concentration. It was also noted during the course of analysis that when the residence time selected is high the longer it takes for the reaction to reach completion, but the smaller the residence time chosen, the faster the reaction, thus, saves time. The same also applies when the mixing parameter is large the faster it is for the reaction to reach completion but then considering a large reactor tank this might cause a breakdown of the impeller.

The recovered class increases due to individuals who came from treatment at the rate η, τ and diminishes as a result of death occurred naturally at a rate μ . The human population experience natural death rate. The bacterial population in contaminated foods and waters where an infectives can contribute to increase the number of bacterial in foods and waters without proper sanitation with a discharge rate of e where c_s the rate of compliance with environmental sanitation and μ_b has been considered to be the death of Salmonella bacteria.

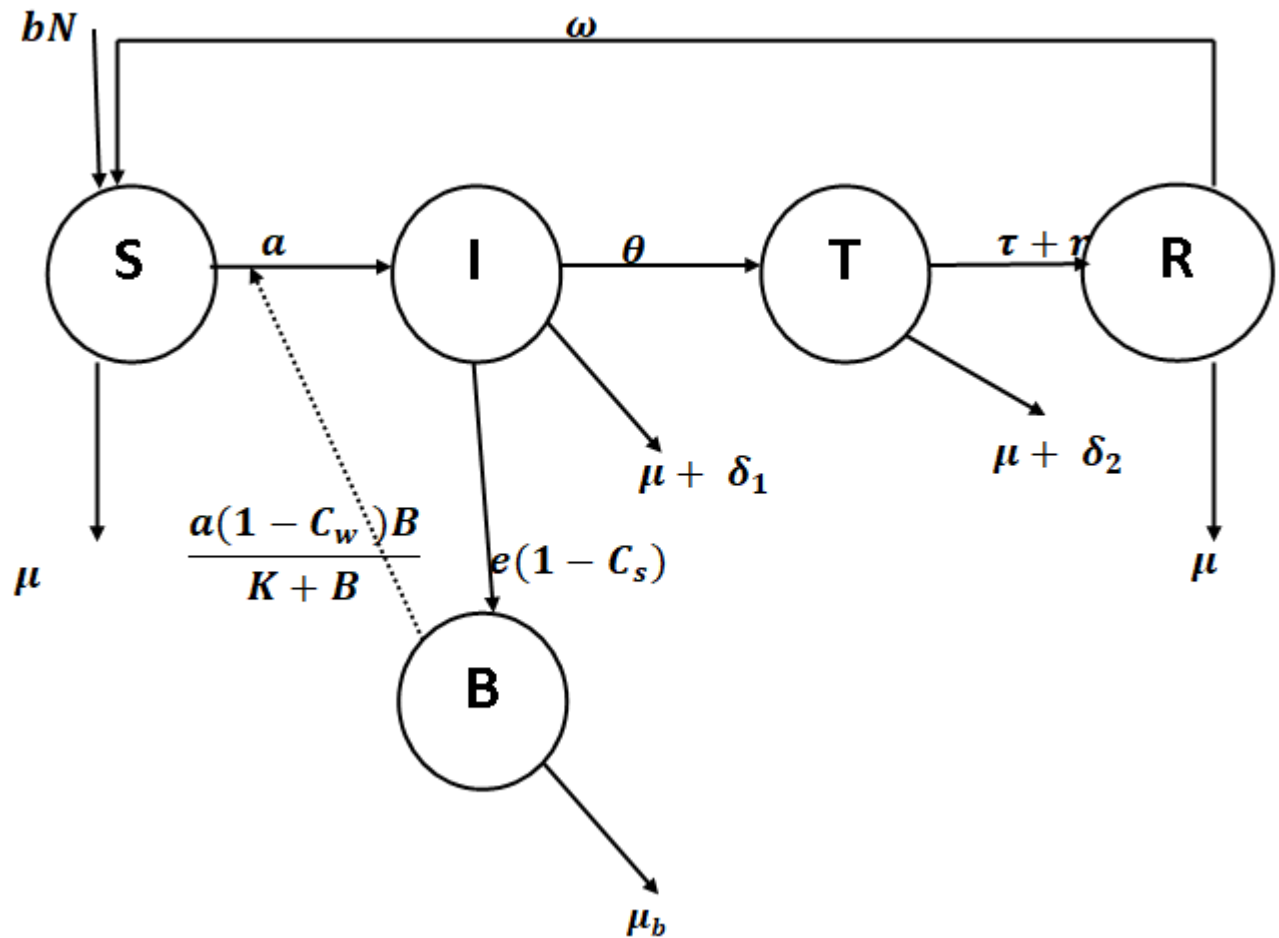


Fig1: Diagrammatic representation of typhoid fever

MODEL EQUATIONS

The Model consists of the following system of ordinary differential equations.

$$\frac{dS}{dt} = bN - \frac{a(1 - c_w)B}{K + B} S + \omega R - \mu S \quad (1)$$

$$\frac{dI}{dt} = \frac{a(1 - c_w)B}{K + B} S - (\theta + \mu + \delta_1) I \quad (2)$$

$$\frac{dT}{dt} = \theta I - (\eta + \tau + \mu + \delta_2) T \quad (3)$$

$$\frac{dR}{dt} = (\eta + \tau) T - (\omega + \mu) R \quad (4)$$

$$\frac{dB}{dt} = e(1 - c_s) I - \mu_b B \quad (5)$$

$$\Omega = \{(S, I, T, R, B) \in \mathbf{R}_+^5 : S + I + T + R = 1, S \geq 0, I \geq 0, T \geq R \geq 0, B \geq 0, \} \quad (6)$$

Which can be shown to be positively invariant with respect to the system (1-5).

Model variables and parameters in the model are listed below

- S Susceptible individuals
- I Infected individuals
- T Treated individuals
- R Recovered individuals
- B Bacterial populations
- N Total number of human population
- b Per capital birth rate.
- μ Per capital normal death rate
- θ Progression rate from I to T
- a Salmonella ingestion rate
- e Discharge rate Salmonella
- K Concentration of salmonella bacterial in food and water
- δ_1 Typhoid induced death rate from infectives

- δ_2 Typhoid induced death rate from treated
- ω Loss rate (waning) of immunity by recovered individuals
- η Natural recovery
- τ Recovery due to treatment

ANALYSIS OF THE MODEL

There is absence of infection at the disease. Free equilibrium state. Therefore, all the infected classes will be equal to zero and the whole population will be made up of only the susceptible individuals. We equate the right hand of the model equation (1)-(5) to zero and evaluating it at $I= T= R = B = 0$ and we solve for the susceptible individuals. Hence, the disease-free equilibrium state E^f is

$$(S^*, I^*, T^*, R^*, B^*) = \left(\frac{bN}{\mu}, 0, 0, 0, 0 \right). \tag{7}$$

This represents the state in which there is no infection and is called disease-free equilibrium state.

Effective Basic Reproduction Number R^c

The threshold parameters is one of the most generally used in the study of the stability of equilibrium which is known as basic reproduction number R_0 . Basic reproduction number, R_0 is defined by (Diekamann and Heesterbeek ,2000) and (Murry ,2002) “ as the average number of new infection introduced into any population of vulnerable by actively infected person. If $R_0 < 1$, disease-free equilibrium is locally asymptotically stable and conversely when $R_0 > 1$, the disease free equilibrium is unstable and disease invasion is possible”. If $R_0 = 1$ is a maximum limit below which the introduction of newly infected person in a particular society makes the infection difficult to maintain (Diekamann et al, 1990).

The next generation operation approach described by (Van Den Driessche and Watmough, 2002) will be applied to established the effective basic reproduction number (R^c).

R^c is the largest eigenvalue (spectral radius ρ) of the next generation matrix, FV^{-1} . I.e $R^c = \rho FV^{-1}$ is computed as follows.

Observing that, the matrixes of F and V are obtained from the coefficients of I and B in the third and fifth equation of the model (1)-(5)

$$\begin{aligned} \frac{dI}{dt} &= \frac{a(1-c_w)B}{K+B} S - (\theta + \mu + \delta_1)I \\ \frac{dB}{dt} &= e(1-c_s)I - \mu_b B \end{aligned}$$

$$F = \begin{pmatrix} 0 & \beta S^* \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} K_1 & 0 \\ -e(1-c_s) & K_4 \end{pmatrix} \tag{8}$$

Where

$$K_1 = (\theta + \mu + \delta_1), K_2 = ((\eta + \tau + \mu + \delta_2)), K_3 = (\omega + \mu), K_4 = \mu_b \tag{9}$$

In order to determine the matrix V^{-1} , we use the adjoint method,

i.e.

$$V^{-1} = \begin{pmatrix} \frac{1}{K_1} & 0 \\ \frac{e(1-c_s)}{K_1 K_4} & \frac{1}{K_4} \end{pmatrix} \tag{10}$$

The product of F and V^{-1} gives

i.e.

$$FV^{-1} = \begin{pmatrix} \frac{\beta S^* e(1-c_s)}{K_1 K_4} & \frac{\beta S^*}{K_4} \\ 0 & 0 \end{pmatrix} \tag{11}$$

Then

$$R_c = (\rho)K = (\rho)FV^{-1}$$

According to Castillo- Chavez et al; (2002), effective basic reproduction number, R^c is the dominant eigen value

$$R_c = \frac{\beta S^* e(1-c_s)}{K_1 K_4} = \frac{aeb(1-c_w)(1-c_s)N}{K\mu(\theta + \mu + \delta_1)\mu_b} \tag{12}$$

Where

$$\beta = \frac{a(1-c_w)}{K}, S^* = \frac{bN}{\mu}, K_1 = (\theta + \mu + \delta_1),$$

and $K_4 = \mu_b$

One of the most important concern parameter for a disease to invade a population is R_C which measure the severity of epidemic.

Theorem 1: The disease-free equilibrium is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$

Analysis of local stability of disease-free equilibrium state E_f

To investigate the stability of disease-free equilibrium state E_f , linearization approach shall be used to prove that the disease- free

equilibrium state is locally asymptotically stable. The jacobian stability approach will be used as explain in (Hethcote, 2000) and (Benya, 2013) to attack model equation (1)-(5).

$$\begin{aligned} bN - \frac{a(1-c_w)B^*}{K+B^*}S^* + \omega R^* - \mu S^* &= 0 \\ \frac{a(1-c_w)B^*}{K+B^*}S^* - (\eta + \tau + \delta + \mu)I^* &= 0 \\ \theta I^* - (\eta + \tau + I + \mu + \delta_2) &= 0 \\ (\eta + \tau)I^* - (\omega + \mu)R^* &= 0 \\ e(1-c_s)I^* - \mu_b B^* &= 0 \end{aligned} \tag{13}$$

Linearization of equation (13) gives

$$J(E^f) = \begin{pmatrix} \frac{-a(1-c_w)B^*}{K+B^*} - \mu & 0 & 0 & \omega - \frac{a(1-c_w)}{K+B^*}S^* \\ \frac{a(1-c_w)B^*}{K+B^*} & -k_1 & 0 & \frac{a(1-c_w)}{K+B^*}S^* \\ 0 & \theta & -k_2 & 0 \\ 0 & 0 & \eta + \tau & -k_3 \\ 0 & e(1-c_s) & 0 & -k_4 \end{pmatrix}$$

Where

$$K_1 = (\theta + \mu + \delta_1), K_2, (\eta + \tau + \mu + \delta_2), K_3 = (\omega + \mu), K_4 = \mu_b \text{ and } \beta = \frac{a(1-c_w)}{K}$$

At disease- free equilibrium state E^f

$$(S^*, I^*, R^*, B^*) = \left(\frac{bN}{\mu}, 0, 0, 0 \right)$$

$$J(E^f) = \begin{pmatrix} -\mu & 0 & 0 & \omega - \beta S^* \\ 0 & -k_1 & 0 & \beta S^* \\ 0 & \theta & -k_2 & 0 \\ 0 & 0 & \eta + \tau & -k_3 \\ 0 & e(1-c_s) & 0 & -k_4 \end{pmatrix}$$

Therefore, we observed that all the eigenvalues are real part and negatives which shows that the model is locally asymptotically stable.

Global Stability of Disease Free Equilibrium, E_f

In order to ensure that typhoid fever is independent of the initial size of the sub-populations of the model (1) – (5), it is necessary to show that the DFE is globally-asymptotically stable (GAS). One common approach in studying the global asymptotic stability of the DFE is to construct an appropriate Lyapunov function (Li et al., 1999, Fall et al., 2007, Huo et al., 2010, Garba and Gumel, 2010). However, we shall apply the result introduced by Castillo-Chavez et al. (2002).

Theorem 2: The disease- free equilibrium E_f of (1)- (5) is globally asymptotically stable (GAS) in Ω if $R_C < 1$.

Proof: To establish the global stability of the disease free equilibrium, the two conditions (H1) and (H2) as in Castillo-Chavez et al. (2002) must be satisfied for $R_C < 1$. We rewrite the model (1) – (5) in the form:

$$\begin{aligned} \frac{dX}{dt} &= F(X, Z) \\ \frac{dZ}{dt} &= G(X, Z) = 0 \end{aligned} \tag{14}$$

Where $X = (S^*, R^*)$ and $Z = (I^*, T^*, B^*)$, with the components of $X \in R^2$ denoting the uninfected population and the components of $Z \in R^3$ denoting the infected population.

The disease-free equilibrium is now denoted as

$$E_f = (X^*, 0), X_1^* = \left(\frac{bN}{\mu}, 0\right)$$

Now, for the first condition, H¹ that is globally asymptotically stability of X^* , we

$$\frac{dX}{dt} = F(X, 0) = \begin{bmatrix} bN + \omega R^* - \mu S^* \\ -(\omega + \mu)R^* \end{bmatrix}$$

have (15)

are linear differential equations, that is

$$\begin{aligned} \frac{dS^*}{dt} &= bN + \omega R^* - \mu S^* \\ \frac{dR^*}{dt} &= -(\omega + \mu)R^* \end{aligned}$$

Solving, we have

$$S^*(t) = \frac{bN + \omega R^*}{\mu} - \frac{bN + \omega R^*}{\mu} e^{-\mu t} + S^*(0)e^{-\mu t}$$

and

$$R^*(t) = R^*(0)e^{-(\omega + \mu)t}$$

Now, clearly from (4) and (7) we have $S^*(t) + R^*(t) \rightarrow N$ as $t \rightarrow \infty$, regardless of the value of $S^*(0)$ and $R^*(0)$.

Thus $X^* = \left(\frac{bN}{\mu}, 0\right)$ is globally asymptotically stable.

Next, for the second condition, that is $\tilde{G}(X, Z) = AZ - G(Z)$, we have

$$G(X, Z) = \begin{bmatrix} G_1(X, Z) \\ G_2(X, Z) \\ G_3(X, Z) \end{bmatrix} = \begin{bmatrix} \frac{a(1-c_w)B^*S^*}{K+B^*} - K_2I^* \\ \xi I^* - K_3T^* \\ e(1-c_s)I^* - K_5B^* \end{bmatrix}$$

$$(A) = \frac{\partial G}{\partial Z}(X^*, 0) = \begin{bmatrix} -k_2 & 0 & \frac{a(1-c_w)S^*}{K+B^*} \\ \xi & -k_3 & 0 \\ e(1-c_s) & 0 & -k_5 \end{bmatrix}$$

This is clearly an M-matrix (the off-diagonal elements of A are non-negative).

So that

$$\tilde{G}(X, Z) = AZ - G(X, Z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $E^f = (X^*, 0)$ is globally asymptotically stable. This shows that irrespective of initial population of an infected, typhoid fever can still be controlled.

Conclusion

In this paper, we developed and analyzed a compartmental model for the transmission dynamic of typhoid fever. In the present study, we used SITR-B epidemic model to study the influence of control measures. In the model we introduced some control measures which can help in reducing the burden of the disease.

The effective basic reproduction number R^c is obtained. Our analysis shows that the disease-free equilibrium of the model is locally and globally stable when $R^c < 1$ and unstable when $R^c > 1$.

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